

Entanglement Generation by Electric Field Background

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The quantum vacuum is unstable under the influence of an external electric field and decays into pairs of charged particles, a process which is known as the Schwinger pair production. We propose and demonstrate that this electric field can generate entanglement. Using the Schwinger pair production for constant and pulsed electric fields, we study entanglement for scalar particles with zero spins and Dirac fermions. One can observe the variation of the entanglement produced for bosonic and fermionic modes with respect to different parameters.

PACS numbers: 03.67.Bg, 03.67.-a, 04.62.+v

I. INTRODUCTION

The quantum information theory is important for the multitude of its promising new applications in such varied fields as quantum communication and teleportation, quantum cryptography, quantum computing, etc. The concept of entanglement also plays crucial roles in black hole thermodynamics [1, 2] and in the information loss problem [3–5], which have given rise to many studies aimed at measuring the generation and degradation of entanglement in a wide spectrum of systems. These studies include investigation of entanglement in both inertial [6] and non-inertial frames [7–12] as well as its generation in expanding spacetime [13, 14] and in relativistic quantum fields [15].

Although many of these works are far from being experimental, they are valuable as they offer a refined understanding of quantum information. In this paper, we explore the generation of entanglement using Schwinger pair production. For this purpose, we will investigate the effect of background electric field on the generation of entanglement for scalar and spinor fields.

It is well known that when an external electric field is applied to the quantum electrodynamical vacuum, the vacuum becomes unstable and decays into pairs of charged particles. In fact, the quantum vacuum is unstable under the influence of an external electric field, as the virtual electron-positron dipole pairs gain energy from the external field. When the field is sufficiently strong, these virtual pair particles gain the threshold pair creation energy and become real pairs. This remarkable phenomenon was first predicted by F. Sauter [16] to be later refined by W. Heisenberg and H. Euler [17] and formalized in the language of QED by Schwinger [18], hence its designation nowadays as the Schwinger pair production effect. This phenomenon has been investigated by scholars and workers from a variety of fields [19, 20]. Efforts in the 1900s and early 21st century aimed at descriptions of more realistic field configurations led to the development of different formalisms such as the quantum kinetic approach which were used for the numerical computation of the Schwinger effect [21]. Other approaches used include the closely related scattering-like formalism in terms of the Riccati equation [22], the Dirac-Heisenberg-Wigner formalism [23], and the numerical worldline formalism [24]. The critical electric field required for pair creation is almost 10^{16} V/cm which is too enormous to be directly observed. However, the feasibility of its experimental realization in ultra-intense laser field system [25, 26] has recently led to a re-thinking of the Schwinger effect. It has been realized that the Schwinger limit laser intensity of $4 \times 10^{29} \text{ W/cm}^2$ is not necessarily a strict limit and might be lowered by several orders of magnitude through manipulating the form of the laser pulses [27–31]. Furthermore, it has been proposed that the Schwinger pair production effect may be observed in graphene [32]. These considerations motivated the authors to study the generation of entanglement using an electric field.

The present paper is organized as follows. In Section II, we utilize the Schwinger effect for scalar particles with zero spin and Dirac fermions in the presence of a constant electric field. We will demonstrate that a constant electric field can generate the entanglement that its value can be determined. We will also consider the variation of the entanglement produced for bosonic and fermionic modes with respect to different parameters. In Section III, we extend our investigation to the pulsed electric field. Finally, conclusions will be presented in Section VI.

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II. ENTANGLEMENT GENERATION IN A CONSTANT ELECTRIC FIELD

The Minkowski vacuum becomes unstable by a strong electric field and decays into pairs of charged particles. One can use the 'in' and 'out' formalism in order to investigate the entanglement generation. 'In' and 'out' are related to asymptotic times $t = -\infty$ and $t = +\infty$, respectively. If the separable 'in' state can be expanded in terms of the entangled 'out' state, the generated entanglement can then be determined. The state of two particles A and B is a vector in a $(d \times d')$ -dimensional Hilbert space $H_{ab} = H_a \otimes H_b$. The space H_{ab} is the tensor product of the subspaces H_a and H_b of each particle. An element of the space H_{ab} is written as $|\Phi\rangle_{ab} = \sum_{i,j} C_{ij} |i\rangle_a \otimes |j\rangle_b$. A state $|\Phi\rangle_{ab} \in H_a \otimes H_b$ is separable if $|\Phi\rangle_{ab} = |\Phi\rangle_a \otimes |\Phi\rangle_b$. An entangled state is a state that is not separable [33].

In the following subsections, we study entanglement entropy for charged scalar and fermion particles in the presence of an electric field.

A. Entanglement Entropy For Scalar Particles

In the study of entanglement generation, we use asymptotic solutions of equation of motion for charged scalar particles in the presence of an electric field.

Consider an electric field along the z-direction. It is related to the gauge potential through $E_z(t) = -\partial A_z(t)/\partial t$. For a scalar particle of mass m and Charge q , the Klein-Gordon equation on the four dimensional Minkowski spacetime with the metric $(+, -, -, -)$ is given by

$$[(\partial_\mu - iqA_\mu)(\partial^\mu - iqA^\mu) + m^2]\phi(t, x) = 0, \quad (1)$$

where, $A_\mu = (0, 0, 0, A_z(t))$ and ϕ is the scalar field.

For the purpose of the present subsection, we restrict ourselves to the constant electric field and rewrite Eq. (1) for $E_z(t) = E_0$:

$$[\partial_t^2 + m^2 + \hat{k}_\perp^2 + (\hat{k}_z - qE_0t)^2]\phi_{ks}(t, x) = 0 \quad (2)$$

In the above equation, natural units are used in which $c = \hbar = 1$ and $A_z(t)$ is replaced by $A_z(t) = -E_0t$. This equation is used for scalar particles with zero spin. After turning to the momentum space, we have

$$\phi(t, r) = (2\pi)^{-3/2} \int dk \exp(ik \cdot r) \tilde{\phi}(t, k). \quad (3)$$

Therefore, Eq. (2) in the momentum space is given by

$$[\partial_t^2 + m^2 + k_\perp^2 + (k_z - qE_0t)^2]\tilde{\phi}_k(t) = 0 \quad (4)$$

where, $\tilde{\phi}_k(t) = \tilde{\phi}(t, k)$ is the Fourier component of the Klein-Gordon equation for scalar particles and $k_\perp^2 = k_x^2 + k_y^2$. Changing to the following convenient variables

$$\begin{aligned} z &= \sqrt{2}\xi e^{i\pi/4}, \quad \xi = \frac{(k_z - qE_0t)}{\sqrt{qE_0}}, \\ \nu &= -\frac{1}{2} - i\frac{\mu}{2}, \quad \mu = \frac{m^2 + k_\perp^2}{qE_0}, \end{aligned} \quad (5)$$

Eq. (4) will be converted to the following equation:

$$[\partial_z^2 + (\nu + \frac{1}{2} - \frac{z^2}{4})]\tilde{\phi}_\nu(z) = 0. \quad (6)$$

The solutions of Eq. (6) are the parabolic cylinder functions denoted by the symbol $D_\nu(z)$ [34]

$$D_\nu(z) = 2^{\frac{\nu-1}{2}} e^{-\frac{z^2}{4}} \Psi(\frac{1-\nu}{2}, \frac{3}{2}, \frac{z^2}{2}) \quad (7)$$

where, $\Psi(a, b, z)$ is the confluent hypergeometric function. The functions $D_\nu(-z)$ and $D_{-\nu-1}(\pm iz)$ also satisfy Eq. (6) [34]. The following linear relations between parabolic cylinder functions show how any three of the solutions are

connected:

$$\begin{aligned} D_\nu(z) &= \frac{\Gamma(\nu+1)}{2\pi} [e^{\pi/2} D_{-\nu-1}(iz) + e^{-\frac{i\pi\nu}{2}} D_{-\nu-1}(-iz)] \\ &= \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{-\frac{i\pi(\nu+1)}{2}} D_{-\nu-1}(iz) + e^{-i\pi\nu} D_\nu(-z). \end{aligned} \quad (8)$$

Therefore, there are precisely two linearly independent solutions of Eq. (6). For all values of ν , $D_\nu(z)$ and $D_{-\nu-1}(\pm iz)$ are linearly independent. In order to calculate entanglement, we need the asymptotic solutions at $t_{in} \rightarrow -\infty$ and $t_{out} \rightarrow +\infty$ because we are interested in solutions with negative and positive frequencies. The asymptotic behavior of the solutions for large values of $|z|$ is given by [34]

$$D_\nu(z) \approx e^{-z^2/4} z^\nu, (|z| \gg |\nu|, |\arg(z)| < \frac{3\pi}{4}) \quad (9)$$

Using Eqs. (6) and (7), one can find the asymptotic solution at $t_{in} = -\infty$ for a particle with momentum k and charge q

$$\begin{aligned} D_\nu(z) &= D_{-\frac{1}{2}-i\frac{\mu}{2}}(\sqrt{2}\xi e^{i\pi/4}) \\ &\approx (2\xi^2)^{(-i\mu-1)/4} e^{(\mu-i)\pi/8} e^{-i\xi^2/2}, \end{aligned} \quad (10)$$

where, $\xi \gg 1$. Using $k \rightarrow -k$ and $q \rightarrow -q$ in Eq. (4), we obtain another solution with negative frequency which describes an incoming antiparticle as below:

$$\begin{aligned} D_{-\nu-1}(-iz) &= D_{-\frac{1}{2}+i\frac{\mu}{2}}(\sqrt{2}\xi e^{-i\pi/4}) \\ &\approx (2\xi^2)^{(i\mu-1)/4} e^{(\mu+i)\pi/8} e^{i\xi^2/2}. \end{aligned} \quad (11)$$

In these solutions, the asymptotic phases and frequencies are:

$$\begin{aligned} \pm \frac{1}{2}\xi^2 &= \frac{1}{2qE_0} k_z^2 - k_z t + \frac{1}{2} qE_0 t^2, \\ \pm \partial_t \frac{1}{2}\xi^2 &= \pm(-k_z + qE_0 t) \sim \pm\omega. \end{aligned} \quad (12)$$

We can also find the sets of solutions at $t_{out} = +\infty$. For an outgoing particle with momentum k and charge q , the convenient solution is

$$D_{-\nu-1}(iz) = D_{-\frac{1}{2}-i\frac{\mu}{2}}(\sqrt{2}|\xi| e^{-i\pi/4}), \quad (13)$$

where, $|\xi| = \frac{-k_z + qE_0 t}{\sqrt{qE_0}} \gg 1$. In the same manner, $D_\nu(-z)$ describes an outgoing antiparticle. Using Bogoliubov transformation [35], one can expand the sets of solutions at $t_{in} = -\infty$ in terms of the sets of solutions at $t_{out} = +\infty$ as follows:

$$\phi_{in,k}^+ = \alpha_k \phi_{out,k}^+ + \beta_k \phi_{out,k}^-, \quad (14)$$

where, α_k and β_k are Bogoliubov coefficients. The 'in' positive frequency mode $\phi_{in,k}^+ = D_{\nu,in}(z)$ is expressed as a linear combination of the 'out' positive $\phi_{out,k}^+ = D_{-\nu-1,out}(iz)$ and negative $\phi_{out,k}^- = D_{\nu,out}(-z)$ frequency modes. Using a linear relation between $D_\nu(\pm z)$ and $D_{-\nu-1}(\pm iz)$, Eq. (8), one can achieve the Bogoliubov coefficients as follows:

$$\alpha_k = \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{-\frac{i\pi(\nu+1)}{2}}, \quad \beta_k = e^{-i\pi\nu}. \quad (15)$$

Taking into account [34]

$$\frac{\pi}{|\Gamma(\frac{1}{2} + ix)|^2} = \cosh(\pi x), \quad (16)$$

these coefficients for scalar particles will satisfy the following relation

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \quad (17)$$

Now, we calculate the entanglement which is generated by the background constant electric field. It is necessary to specify the 'in' and 'out' states and operators. The operators $a_{k,in}$, $b_{k,in}$ and $a_{k,out}$, $b_{k,out}$ annihilate the 'in' $|0_k 0_{-k}\rangle_{in}$ and 'out' $|0_k 0_{-k}\rangle_{out}$ vacuum for each momentum, respectively.

$$\begin{aligned} a_{k,in} |0_k 0_{-k}\rangle_{in} &= b_{k,in} |0_k 0_{-k}\rangle_{in} = 0 \\ a_{k,out} |0_k 0_{-k}\rangle_{out} &= b_{k,out} |0_k 0_{-k}\rangle_{out} = 0 \end{aligned} \quad (18)$$

where, the (k,-k) subscripts indicate the particle and antiparticle modes. Using the Bogoliubov transformation, the relation between these operators is given by [35]

$$\begin{aligned} a_{k,in} &= \alpha_k^* a_{k,out} - \beta_k^* b_{k,out}^\dagger \\ b_{k,in} &= \alpha_k^* b_{k,out} - \beta_k^* a_{k,out}^\dagger \end{aligned} \quad (19)$$

Now, using the convenient calculations, we show that the separable 'in' states can be expanded in terms of the 'out' entangled state. The state vector of the system can be described by the tensor product of the two Hilbert spaces $H_k \otimes H_{k'}$, where H_k indicates the Hilbert space related to particles and $H_{k'}$ to the antiparticles created by the electric field. The *in*-vacuum state is defined by the absence of any mode excitations

$$|0\rangle_{in} = \prod_{kk'} (|0_k\rangle |0_{k'}\rangle)_{in},$$

Using the Schmidt decomposition, the in-vacuum state for each mode can be expanded in terms of the out-states [36]

$$(|0_k\rangle |0_{-k}\rangle)_{in} = \sum_n c_n (|n_k\rangle |n_{-k}\rangle)_{out},$$

where, n indicates the number of particles with momentum k and the number of antiparticles with momentum $-k$ created by the electric field. For simplicity, $|n_k\rangle |n_{-k}\rangle$ is replaced by $|n_k n_{-k}\rangle$, and we will, therefore, have

$$|0_k 0_{-k}\rangle_{in} = \sum_n c_n |n_k n_{-k}\rangle_{out}. \quad (20)$$

The $|0_k 0_{-k}\rangle_{in}$ is a separable state from the view point of an observer in the *in*-region. If there are more than one non-zero coefficients on the right hand side of Eq.(20), then the separable *in*-state is the entangled state from the view point of an inertial observer in the *out*-region. Therefore, we have to determine c_n to evaluate the measure of the entanglement. For this purpose, we use the definition of vacuum and its normalization. Substituting Eqs. (19) and (20) in the definition of vacuum

$$\begin{aligned} a_{k,in} |0_k 0_{-k}\rangle_{in} &= 0 \\ (\alpha_k^* a_{k,out} - \beta_k^* b_{k,out}^\dagger) \sum_n c_n |n_k n_{-k}\rangle_{out} &= 0, \end{aligned} \quad (21)$$

leads to

$$c_{n+1} = \frac{\beta_k^*}{\alpha_k^*} c_n, \quad (22)$$

Normalization of vacuum,

$$\langle 0_k 0_{-k} | 0_k 0_{-k} \rangle_{in} = 1$$

leads to

$$\begin{aligned} \sum_n |c_n|^2 &= 1 \\ &= |c_0|^2 (1 + |\frac{\beta_k}{\alpha_k}|^2 + |\frac{\beta_k}{\alpha_k}|^4 + \dots). \end{aligned} \quad (23)$$

Thus, the coefficients c_n are given by:

$$\begin{aligned} |c_0|^2 &= |\frac{1}{\alpha_k}|^2 \\ |c_n|^2 &= |\frac{\beta_k}{\alpha_k}|^{2n} |c_0|^2 \\ &= (1 - |c_0|^2)^n |c_0|^2 \end{aligned} \quad (24)$$

Based on the values of c_n thus obtained, we expect an entanglement generation to occur. We can utilize an appropriate measure of entanglement, namely the von Neumann entropy defined as follows:

$$S(\rho_k) = -\text{Tr}(\rho_k \log_2(\rho_k)). \quad (25)$$

First, we have to specify the density matrix of the whole system, $\rho_{k,-k}$, followed by reduced density matrix of the subsystem, ρ_k . All the properties of the system can be deduced from the density matrix

$$\begin{aligned} \rho_{k,-k} &= |0_k 0_{-k}\rangle_{in} \langle 0_k 0_{-k}| \\ &= \sum_{n,m} c_n c_m^* |n_k n_{-k}\rangle_{out} \langle m_k m_{-k}|. \end{aligned} \quad (26)$$

As we wish to deal with only one of the subsystems, we use the concept of reduced density matrix. One can find the reduced density matrix for the subsystem related to the particles (denoted by k), obtained by tracing $\rho_{k,-k}$ over all the states of the subsystem related to the antiparticles (denoted by $-k$), so that

$$\begin{aligned} \rho_k &= \text{Tr}_{-k}(\rho_{k,-k}) \\ &= \sum_l \langle l_{-k} | \rho_{k,-k} | l_{-k} \rangle \\ &= \sum_n |c_n|^2 |n_k\rangle \langle n_k|. \end{aligned} \quad (27)$$

The von Neumann entropy for scalar modes described by Eq. (25) is given by

$$\begin{aligned} S_k &= -\sum_n |c_n|^2 \log_2 |c_n|^2 \\ &= \log_2 \frac{x^{\frac{x}{x-1}}}{1-x}, \end{aligned} \quad (28)$$

where, $x = |\frac{\beta_k}{\alpha_k}|^2$ and is determined by Eq. (15)

$$x = \frac{|\beta_k|^2}{1 + |\beta_k|^2}, \quad |\beta_k|^2 = e^{\frac{-\pi(m^2 + k_\perp^2)}{qE_0}}. \quad (29)$$

Therefore, we get the von Neumann entropy with respect to the electric field, transverse components of momentum, as well as particle's mass and charge. Eq. (28) can be written in terms of $|\beta_k|^2$

$$S_k = -|\beta_k|^2 \log_2 |\beta_k|^2 + (1 + |\beta_k|^2) \log_2 (1 + |\beta_k|^2). \quad (30)$$

According to Eq. (30), the increase in $|\beta_k|^2$ value enhances the von Neumann entropy. Both the von Neumann entropy, S_k , and $|\beta_k|^2$ are increasing functions with respect to E_0 . The variation of the S_k , and $|\beta_k|^2$ as a function of the electric field E_0 is shown in Fig. 1. In the large electric fields, $|\beta_k|^2$ tends to its maximum value, $|\beta_k|^2 \rightarrow 1$. Thus, regarding Eq. (30), entropy is a function of $|\beta_k|^2$ and at large values of the electric fields E_0 tends to a constant value ($S_k = 2$).

$|\beta_k|^2$ is the mean number of the particles (antiparticles) produced in mode k ($-k$)

$$\langle 0_k 0_{-k} | a_{out,k}^\dagger a_{out,k} | 0_k 0_{-k} \rangle_{in} = |\beta_k|^2. \quad (31)$$

When the mean number of the produced pairs increases, the entanglement generated for bosonic modes will also increase. S_k and $|\beta_k|^2$ exhibit similar behaviors for bosonic modes. The variation of entanglement with respect to mass is shown in Fig. 2. A specific electric field creates more particles of smaller mass than those of larger mass. Fig. 2 indicates that the measure of entanglement for fixed values of q , k_\perp and E_0 is greater for particles with smaller mass than it is for those of larger mass. The maximum value of entanglement for fixed values of q , k_\perp and E_0 occurs in $m = 0$ which is obtained by substituting $|\beta_k|^2 = \exp(\frac{-\pi k_\perp^2}{2qE_0})$ in Eq. (30). Since m and k_\perp appear in the same form in S_k , the entanglement behavior will be similar with respect to k_\perp and m .

In Eq. (20), we express *in*-vacuum in terms of *out*-states. The probability of vacuum-to-vacuum transition is given by

$$|\langle 0, out | 0, in \rangle|^2 = |c_0|^2 = \frac{1}{|\alpha_k|^2}. \quad (32)$$

The maximum value of $|c_0|^2 = 1$; this means that the vacuums of the '*in*' and '*out*' regions are the same. Decreasing value of $|c_0|^2$ means that the initial vacuum decays to the more pairs in the '*out*' region. Therefore, it is reasonable to suggest that a smaller value of $|c_0|^2$ leads to a more entangled state. Since the value of $|\beta|^2$ ranges between 0 and 1, and also because $|\alpha_k|^2 = 1 + |\beta|^2$, the minimum value of $|c_0|^2$ occurs at $|\beta|^2 = 1$.

B. Entanglement Entropy for Fermion Particles

In this subsection, we will investigate the generation of entanglement for fermionic modes. We use asymptotic solution of equation of motion for charged fermion particles in the presence of an electric field. Consider an electric field along the z -direction. It is related to the gauge potential through $E_z(t) = -\partial A_z(t)/\partial t$. For a particle of mass m and charge q , the Dirac equation on the four dimensional Minkowski spacetime with the metric $(+, -, -, -)$ is given by

$$(i\gamma^\mu \partial_\mu - q\gamma^\mu A_\mu - m)\Psi(x) = 0 \quad (33)$$

where, $A_\mu = (0, 0, 0, A_z(t))$. γ^μ and Ψ are the Dirac matrix and spinors, respectively [37]. One can introduce $\Psi(x)$ to have:

$$\Psi(x) = (i\gamma^\nu \partial_\nu - q\gamma^\nu A_\nu + m)R(x) \quad (34)$$

The second order differential equation is

$$\begin{aligned} & (i\gamma^\mu \partial_\mu - q\gamma^\mu A_\mu - m)(i\gamma^\nu \partial_\nu - q\gamma^\nu A_\nu + m)R \\ &= [-\partial_t^2 - m^2 - \partial_x^2 - \partial_y^2 - (\partial_z - iqA_z)^2 + iqE_z\alpha^3]R \end{aligned} \quad (35)$$

where, $\alpha^3 = \gamma^0\gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$. We search for a solution of Eq. (35) of the following form:

$$\begin{aligned} R(x) &= \varphi(t, \mathbf{x})\chi \\ \varphi(t, \mathbf{x}) &= (2\pi)^{-3/2} \int e^{i\mathbf{k}\cdot\mathbf{x}} \phi(t, \mathbf{k}) d\mathbf{k} \end{aligned} \quad (36)$$

where, $\varphi(x)$ is a complex scalar function and χ designates the eigenbispinors of the α^3 and $\Sigma_3 = \frac{i}{2}\gamma^1\gamma^2$:

$$\begin{aligned} \chi_+^\uparrow &= \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \chi_+^\downarrow = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \\ \chi_-^\uparrow &= \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \chi_-^\downarrow = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \\ \alpha^3\chi_\pm^s &= \chi_\pm^s, \quad (s \in \{\uparrow, \downarrow\}) \\ \Sigma_3\chi_\pm^\uparrow &= +\chi_\pm^\uparrow, \quad \Sigma_3\chi_\pm^\downarrow = -\chi_\pm^\downarrow, \end{aligned} \quad (37)$$

Σ_3 is the matrix of the spin component along the direction of the electric field and commutes with α^3 .

Using Eqs. (35-37) and substituting the standard representation for the Dirac matrix, we have

$$[\partial_t^2 + m^2 + k_\perp^2 + (k_z + qA_z(t))^2 + iqE(t)]\phi_{k,s}(t)\chi_+^s = 0 \quad (38)$$

where, $\phi_{k,s}(t)\chi_+^s$ and $\phi_{k,s}^*(t)\chi_+^s$ specify the spin-up and down particle and antiparticle, respectively. Then, the solutions of Eq. (38) form a complete set. Another solution of the second order differential equation (35) with negative eigenvalue, $-iqE(t)$, satisfies the following equation

$$[\partial_t^2 + m^2 + k_\perp^2 + (k_z + qA_z(t))^2 - iqE(t)]\phi_{k,s}(t)\chi_-^s = 0 \quad (39)$$

The second order differential equation (35) leads to Eqs.(38) and (39), while the Dirac equation is a first order one. Therefore, it will suffice to have one complete set of solutions corresponding to either (38) or (39). In fact, Eq. (39) does not lead to any new result. Therefore, we consider Eq. (38) and write R in the following form

$$R(t, k) = \sum_s (c_1^s \phi_{k,s}(t)\chi_+^s + c_2^s \phi_{k,s}^*(t)\chi_+^s) \quad (40)$$

Using (34), Ψ takes the following form

$$\Psi(t, k) = \sum_{k,s} (a_{k,s} u_k^s(x) + b_{k,s}^\dagger v_k^s(x)) \quad (41)$$

with

$$\begin{aligned} u_k^s(t) &= (i\gamma^0 \partial_t - \vec{\gamma} \cdot (\vec{k} + q\vec{A}) + m) \phi(t, k) \chi_+^s \\ v_k^s(t) &= (i\gamma^0 \partial_t - \vec{\gamma} \cdot (\vec{k} + q\vec{A}) + m) \phi(t, k)^* \chi_+^s \end{aligned} \quad (42)$$

According to Eq. (6), $\phi_{k,s}(t)$ in Eq. (38) are parabolic cylinder functions

$$\begin{aligned} D_\nu(\pm z), D_{-\nu-1}(\pm iz) \\ \nu = -1 - i\frac{\mu}{2}, \quad \mu = \frac{m^2 + k_\perp^2}{qE}. \end{aligned} \quad (43)$$

Using Eqs. (42) and (43) and the invariant inner product

$$(f_k^r(t, x), g_p^s(t, x)) = \int (f_k^r(t, x))^\dagger g_p^s(t, x) d^3x \quad (44)$$

one can evaluate the Bogoliubov coefficients for Dirac's fermions in a background constant electric field as follows

$$\begin{aligned} (u_k^{s,in}, u_p^{r,out}) &= \delta_{rs} \delta(\vec{k} - \vec{p}) \alpha_k^s \\ (u_k^{s,in}, v_p^{r,out}) &= \delta_{rs} \delta(\vec{k} - \vec{p}) \beta_k^s, \end{aligned} \quad (45)$$

with

$$\begin{aligned} \alpha_k^s &= \alpha_k^\uparrow = \alpha_k^\downarrow = \sqrt{\frac{\mu}{\pi}} \Gamma\left(\frac{i\mu}{2}\right) \sinh\left(\frac{\pi\mu}{2}\right) e^{-\frac{\pi\mu}{4}} \\ \beta_k^s &= \beta_k^\uparrow = \beta_k^\downarrow = e^{-\pi\frac{\mu}{2}}. \end{aligned} \quad (46)$$

As indicated in Eq. (37), $s \in \{\uparrow, \downarrow\}$ is related to the positive and negative eigenvalues of the matrix of the spin component along the direction of the electric field. Since the spin has no interaction with the electric field, the Bogoliubov coefficients for the up and down spins are the same.

Taking [34] into account

$$|\Gamma(ix)|^2 = \frac{\pi}{x \sinh \pi x} \quad (47)$$

these coefficients will satisfy the relation below

$$|\alpha|^2 + |\beta|^2 = 1 \quad (48)$$

The relationship between the 'in' and the 'out' operators is expressed by

$$\begin{aligned} a_{d,out} &= \alpha_d a_{d,in} - \beta_d^* b_{d,in}^\dagger \\ b_{d,out}^\dagger &= \alpha_d^* b_{d,in}^\dagger + \beta_d a_{d,in}, \end{aligned} \quad (49)$$

where, a_d and b_d are the annihilation operators for particle and antiparticle, respectively, and subscript d stands for momentum k and spin $s \in \{\uparrow, \downarrow\}$. The vacuum state is given by

$$|0\rangle_{in} = \prod_{k,k',s} (|0_k^s\rangle |0_{k'}^s\rangle)_{in}. \quad (50)$$

Using the Schmidt decomposition and Pauli exclusion principle, we can expand the 'in' vacuum state in terms of the 'out' states for a single mode k

$$\begin{aligned} |0_k 0_{-k}\rangle_{in} &\equiv (|0_k^\uparrow 0_{-k}^\downarrow\rangle |0_k^\downarrow 0_{-k}^\uparrow\rangle)_{in} = \sum_{n=0,1} c'_n |n_k^\uparrow n_{-k}^\downarrow\rangle_{out} \sum_{m=0,1} c''_m |m_k^\downarrow m_{-k}^\uparrow\rangle_{out} \\ &= c_0 |0_k^\uparrow 0_{-k}^\downarrow\rangle_{out} |0_k^\downarrow 0_{-k}^\uparrow\rangle_{out} + c_1 |1_k^\uparrow 1_{-k}^\downarrow\rangle_{out} |1_k^\downarrow 1_{-k}^\uparrow\rangle_{out} \\ &+ c_2 |1_k^\uparrow 1_{-k}^\downarrow\rangle_{out} |0_k^\downarrow 0_{-k}^\uparrow\rangle_{out} + c_3 |0_k^\uparrow 0_{-k}^\downarrow\rangle_{out} |1_k^\downarrow 1_{-k}^\uparrow\rangle_{out}, \end{aligned} \quad (51)$$

where, the coefficients $c'_n c''_m$ are replaced to c_i and symbol \uparrow (\downarrow) indicate up(down) spin. Imposing $a_{d,in}|0_k 0_{-k}\rangle_{in} = 0$, $(\langle 0_k 0_{-k}|0_k 0_{-k}\rangle)_{in} = 1$ and using Eq. (49), we obtain the four coefficients c_i

$$\begin{aligned} |c_0|^2 &= |\alpha_k^\uparrow|^2 |\alpha_k^\downarrow|^2, & |c_1|^2 &= |\beta_k^\uparrow|^2 |\beta_k^\downarrow|^2 \\ |c_2|^2 &= |\alpha_k^\uparrow|^2 |\beta_k^\downarrow|^2, & |c_3|^2 &= |\alpha_k^\downarrow|^2 |\beta_k^\uparrow|^2. \end{aligned} \quad (52)$$

The states in (51) are designated by A, B, C and D.

$$\begin{aligned} |0_A\rangle_{out} &\equiv |0_k^\uparrow\rangle_{out}, & |1_A\rangle_{out} &\equiv |1_k^\uparrow\rangle_{out}, \\ |0_B\rangle_{out} &\equiv |0_k^\downarrow\rangle_{out}, & |1_B\rangle_{out} &\equiv |1_k^\downarrow\rangle_{out}, \\ |0_C\rangle_{out} &\equiv |0_{-k}^\uparrow\rangle_{out}, & |1_C\rangle_{out} &\equiv |1_{-k}^\uparrow\rangle_{out}, \\ |0_D\rangle_{out} &\equiv |0_{-k}^\downarrow\rangle_{out}, & |1_D\rangle_{out} &\equiv |1_{-k}^\downarrow\rangle_{out}, \end{aligned} \quad (53)$$

Using the representation (53), the 'in' vacuum state is expressed by

$$(|0_k^\uparrow 0_{-k}^\downarrow\rangle |0_k^\downarrow 0_{-k}^\uparrow\rangle)_{in} = |\Psi_{ABCD}\rangle_{out}. \quad (54)$$

We can calculate the measure of entanglement between the one part of system to the rest.

$S_{A(BCD)}$, for example, is the measure of entanglement between the state A and the states B, C, D as follows

$$S_{A(BCD)} = -Tr(\rho_A \log_2 \rho_A). \quad (55)$$

The reduced density operator ρ_A in $S_{A(BCD)}$ after tracing on B, C, and D is given by

$$\begin{aligned} \rho_A &= Tr_{(BCD)}(|\Psi_{ABCD}\rangle_{out} \langle \Psi_{ABCD}|) \\ &= (|c_0|^2 + |c_3|^2) |0_A\rangle_{out} \langle 0_A| \\ &\quad + (|c_1|^2 + |c_2|^2) |1_A\rangle_{out} \langle 1_A| \\ &= |\alpha_k|^2 |0_A\rangle_{out} \langle 0_A| + |\beta_k|^2 |1_A\rangle_{out} \langle 1_A|. \end{aligned} \quad (56)$$

$S_{A(BCD)}$, the entanglement entropy between a spin up particle with mode k to the rest of the system is obtained by

$$S_{A(BCD)} = -|\alpha_k|^2 \log_2(|\alpha_k|^2) - |\beta_k|^2 \log_2(|\beta_k|^2). \quad (57)$$

In the same manner, $S_{B(ACD)}$, $S_{C(ABD)}$ and $S_{D(ABC)}$ can be calculated and their values are equal to $S_{A(BCD)}$

$$S_k = -|\alpha_k|^2 \log_2(|\alpha_k|^2) - |\beta_k|^2 \log_2(|\beta_k|^2), \quad |\beta_k|^2 = e^{\frac{-\pi(m^2+k^2)}{2qE_0}}, \quad |\alpha_k|^2 = 1 + |\beta_k|^2 \quad (58)$$

We can also get the average von Neumann entropy [38] as follows

$$S = \frac{1}{4}(S_{A(BCD)} + S_{B(ACD)} + S_{C(ABD)} + S_{D(ABC)}) \quad (59)$$

As expected, all of the entanglement entropies in Eq. (59) have the same value, because the electric field can not distinguish spin up and down particles. In other words, each of the von Neumann entropies in Eq. (59) means the entanglement between one part of the system which is a particle(antiparticle) with mode k and spin s with the rest of the system.

Eqs. (51) and (52) indicate the expanding of *in*- vacuum in terms of 'out' states. Based on (48), $|\alpha_k|^2$ and $|\beta_k|^2$ in fermionic modes range between 0 and 1. If $|\alpha_k|^2$ and $|\beta_k|^2$ have a value of either zero or one, we will have a separable state that leads to zero entanglement. The maximum entanglement occurs when all the coefficients c_n in Eq. (51) are equal and nonzero. Therefore, the entanglement will have its maximum value in $|\alpha_k|^2 = |\beta_k|^2 = \frac{1}{2}$. The behavior of entanglement entropy for fermionic modes is shown in Figs. 3 and 4. In very small or large electric fields the value of $|\beta_k|^2$ tends to 0 or (1), respectively; therefore, the entropy has its minimum value, $S_{min} = 0$, as indicated in Fig. 3. The maximum value of S can be deduced as below

$$\frac{\partial S}{\partial E_0} = 0, \quad (60)$$

And, therefore,

$$E_0 = \frac{\pi(m^2 + k_\perp^2)}{q \ln(2)}, \quad (61)$$

which is equivalent to $|\beta_k|^2 = \frac{1}{2}$. In Fig. 4 the entanglement entropy is obtained by substituting $|\beta_k|^2 = \exp(\frac{-\pi k_\perp^2}{2qE_0})$ in Eq. (58), for massless particles. Large values of the electric fields correspond to large values of $|\beta_k|^2$ and therefore the small value of S_k . According to Eq. (58), for fixed values of E_0 , k_\perp and q , the maximum value of entropy is equal to one for $m^2 = \frac{E_0 q \ln(2)}{\pi} - k_\perp^2$.

III. THE SAUTER-TYPE ELECTRIC FIELD AND THE ENTANGLEMENT GENERATION

In the previous Section, we showed that the constant electric field can generate the entanglement and worked out its variation. Now, one can explore entanglement generation by the pulsed electric field for scalar particles and Dirac fermions.

A. Scalar Particles

Eq. (4) can be rewritten for Sauter-type electric field along the z direction as: $E(t) = E_0 \text{sech}^2(t/\tau)$ [16] in which τ is the width of the electric field. One can choose A_μ as

$$A_\mu = (0, 0, 0, -E_0 \tau \tanh(\frac{t}{\tau})), \quad (62)$$

The Fourier time component of the Klein-Gordon equation for the scalar particle with zero spin satisfies the equation

$$[\partial_t^2 + \omega_k^2(t)]\phi_k(t) = 0, \quad (63)$$

where

$$\omega_k^2(t) = [k_z - qE_0 \tau \tanh(\frac{t}{\tau})]^2 + k_\perp^2 + m^2 \quad (64)$$

As before, we use the asymptotic solutions at $t_{in} = -\infty$ and $t_{out} = \infty$ in order to expand the separable 'in' state in terms of the entangled 'out' state. In the following, the Bogoliubov coefficient which relates the asymptotic solutions to each other is used to obtain the reduced density matrix and the von-Neumann entropy for scalar particles.

Ref. [34] gives two linearly independent solutions of Eq. (63):

$$\begin{aligned} \phi_k(t) = & (z - 1)^{i\tau\omega_{k,out}/2} z^{-i\tau\omega_{k,in}/2} [C_1 F(a, b, c; z) \\ & + C_2 z^{1-c} F(a - c + 1, b - c + 1, 2 - c; z)], \end{aligned} \quad (65)$$

where, F is the hypergeometric function, and

$$\begin{aligned} z &= \frac{1}{2} \tanh(\frac{t}{\tau}) + \frac{1}{2}, \lambda = \sqrt{(qE_0 \tau^2)^2 - \frac{1}{4}}, \\ a &= \frac{1}{2} + \frac{i}{2}(\tau\omega_{k,out} - \tau\omega_{k,in}) - i\lambda, \\ b &= \frac{1}{2} + \frac{i}{2}(\tau\omega_{k,out} - \tau\omega_{k,in}) + i\lambda, \\ c &= 1 - i\tau\omega_{k,in}. \end{aligned} \quad (66)$$

in which, ω_{in} and ω_{out} are the kinetic energies of the field modes at asymptotic times $t_{in} = -\infty$ and $t_{out} = \infty$

$$\begin{aligned} \omega_{k,in} &= \sqrt{(k_z + qE_0 \tau)^2 + k_\perp^2 + m^2}, \\ \omega_{k,out} &= \sqrt{(k_z - qE_0 \tau)^2 + k_\perp^2 + m^2}. \end{aligned} \quad (67)$$

As we know, the asymptotic solutions are related through the Bogoliubov coefficients. Using the properties of the hypergeometric function discussed in more detail in appendix, one can find Bogoliubov coefficients as follows

$$\begin{aligned} |\beta_k|^2 &= \frac{\cosh[\pi\tau(\omega_{out} - \omega_{in})] + \cosh(2\pi\lambda)}{2 \sinh(\pi\tau\omega_{in}) \sinh(\pi\tau\omega_{out})} \\ |\alpha_k|^2 &= \frac{\cosh[\pi\tau(\omega_{out} + \omega_{in})] + \cosh(2\pi\lambda)}{2 \sinh(\pi\tau\omega_{in}) \sinh(\pi\tau\omega_{out})}. \end{aligned} \quad (68)$$

We can use the method described in Section II above to expand the 'in'- vacuum in terms of the 'out' state and specify the coefficient c_n . Below, Eqs. (24-28) and (68) are used to obtain the von Neumann entropy for scalar particles in the background of the Sauter-type electric field, we have

$$\begin{aligned} S &= \log_2 \frac{x^{\frac{x}{x-1}}}{1-x} \\ x &= \frac{|\beta_k|^2}{|\alpha_k|^2} = \frac{\cosh[\pi\tau(\omega_{out} - \omega_{in})] + \cosh(2\pi\lambda)}{\cosh[\pi\tau(\omega_{out} + \omega_{in})] + \cosh(2\pi\lambda)}. \end{aligned} \quad (69)$$

Variation of the entanglement entropy for the Sauter-type electric field with respect to E_0 , k_z , and τ are indicated in Figs. 5-7. According to Fig. 5, for a small value of τ , the entanglement entropy has a local maximum while the behavior of entanglement will be similar to that depicted in Fig 1 for large values of the same parameter. Dependence of S on the longitude momentum, k_z , is indicated in Fig. 6. There is a peak for a small value of τ , while this dependence will be less for large values of τ . As mentioned before, for bosonic modes higher values of $|\beta_k|^2$ lead to a greater entanglement. In other words, the behavior of S is similar to that of $|\beta_k|^2$.

B. Fermion Particles

In this subsection, we explore the variation of the entanglement that is generated by Sauter-type electric field between fermionic modes. Repeating the same procedure for the constant electric field, Eqs. (33-38), one can find the spin diagonal component of the Dirac equation for spinor QED to satisfy the equation

$$\begin{aligned} [\partial_t^2 + [k_z - qE_0\tau \tanh(\frac{t}{\tau})]^2 + k_\perp^2 + m^2 \\ + iqE_0 \sec h^2(t/\tau)]\phi_k(t) = 0, \end{aligned} \quad (70)$$

Two linear solutions of Eq. (70) are given by

$$\begin{aligned} \phi_{k,s}(t) &= (z-1)^{i\tau\omega_{k,out}/2} z^{-i\tau\omega_{k,in}/2} [C_1 F(a, b, c; z) \\ &+ C_2 z^{1-c} F(a-c+1, b-c+1, 2-c; z)], \end{aligned} \quad (71)$$

with

$$\begin{aligned} z &= \frac{1}{2} \tanh(\frac{t}{\tau}) + \frac{1}{2}, \lambda = qE_0\tau^2, \\ a &= \frac{i}{2}(\tau\omega_{k,out} - \tau\omega_{k,in}) \pm i\lambda, \\ b &= 1 + \frac{i}{2}(\tau\omega_{k,out} - \tau\omega_{k,in}) \mp i\lambda, \\ c &= 1 - i\tau\omega_{k,in}. \end{aligned} \quad (72)$$

Using Eqs. (40,42,44,45) and the properties of the hypergeometric function, one can find Bogoliubov coefficients as follows

$$\begin{aligned} |\beta_k^\uparrow|^2 &= |\beta_k^\downarrow|^2 = |\beta_k|^2 = \frac{\cosh(2\pi\lambda) - \cosh[\pi\tau(\omega_{out} - \omega_{in})]}{2 \sinh(\pi\tau\omega_{in}) \sinh(\pi\tau\omega_{out})} \\ |\alpha_k^\uparrow|^2 &= |\alpha_k^\downarrow|^2 = |\alpha_k|^2 = \frac{\cosh[\pi\tau(\omega_{out} + \omega_{in})] - \cosh(2\pi\lambda)}{2 \sinh(\pi\tau\omega_{in}) \sinh(\pi\tau\omega_{out})} \end{aligned} \quad (73)$$

where, \uparrow (\downarrow) indicate the up (down) spins. Exploiting the method described in section II, we may expand the 'in'-vacuum in terms of the 'out' states according to Eq. (51). The coefficient (c_0, \dots, c_3) given by Eq. (52), may now be used to obtain the reduced density matrix and, thereby, the average von-Neumann entropy Eq. (59). In fact, the relations (49-56) remain valid. Using these relations, the entropy of von Neumann is obtained by:

$$S = -|\alpha_k|^2 \log_2(|\alpha_k|^2) - |\beta_k|^2 \log_2(|\beta_k|^2), \quad (74)$$

where, $|\alpha|^2$ and $|\beta|^2$ are specified by (73). The variations of the entanglement with respect to E_0 , τ and k_z are indicated in Figs. 8-10. The behavior of S with respect to m and k_\perp for the pulsed and constant electric fields will be the same. Eq. (74) indicates that the maximum value of the entanglement entropy occurs at $|\beta|^2 = \frac{1}{2}$. The minimum value of S_k also occurs at $|\beta|^2$ is equal to one or zero.

When $\tau \rightarrow \infty$, the pulsed electric field $E(t) = E_0 \text{sech}^2(t/\tau)$ tends to the constant electric field E_0 . Also, in this limit, the Bogoliubov coefficients for bosonic and fermionic modes of Eqs. (68) and (73) tend to be as in Eqs. (15) and (46). Since the generated entanglement is obtained in terms of Bogoliubov coefficients, the behavior of S generated by the pulsed electric field can be observed to tend to the constant electric field for large values of τ .

IV. CONCLUSION

We applied the Schwinger pair production theory to constant and pulsed electric fields on a Minkowski spacetime to demonstrate that the background electric field can generate the entanglement. We worked out the entanglement entropy for scalar particles and Dirac fermions created by the background electric field. The behavior of the entanglement was also depicted in figures with respect to different parameters.

For a constant electric field, the entanglement generated for boson modes as a function of E_0 was found to be an increasing function which tended to a specific constant value in the limit of sufficiently large fields (Fig. 1) but that it monotonically decreased with respect to m and (k_\perp) (Fig. 2). For the fermionic mode, however, it was found to be very different. Optimal values of E_0 , m and k_\perp were observed for which the entanglement entropy would be maximized (Figs. 3 and 4).

In the case of a pulsed electric field, the behavior of the entanglement generated by bosonic and fermionic modes with respect to m and k_\perp was observed to be similar to that in the case of the constant electric field. However, for small values of τ , the bosonic entanglement as a function of E_0 was seen to have a local maximum (Fig. 5). High power laser pulses make a good candidate for generating entangled states experimentally.

The authors intend to explore the generation of entanglement by other forms of electric and magnetic fields in future. As another interesting area of research is to study entanglement generation by background electromagnetic fields at finite temperature.

V. APPENDIX A: HYPERGEOMETRIC FUNCTION AND BOGOLIUBOV COEFFICIENTS

The hypergeometric function, $F(a, b, c, z)$, appears in 24 different forms. These solutions are called Kummer's series and can be arranged in six sets such that the four series belonging to each set represent the same function. Any three of these are connected by a linear relation with constant coefficients [34]. We choose appropriate solutions among the six sets for $t = +\infty$ and $t = -\infty$. As we know, the asymptotic solutions are related through the Bogoliubov coefficients

$$\phi_{k,in} = \alpha_k \phi_{k,out} + \beta_k \phi_{k,out}^*. \quad (75)$$

Using the behavior of the hypergeometric function, we can find the Bogoliubov coefficients. According to (66), the value of z at $t = -\infty(+\infty)$ is $0(1)$. Therefore, the hypergeometric functions can be appropriately expressed in terms of z and $1 - z$.

Using the linear relations between the hypergeometric functions [34],

$$F(a, b, c; z) = \lambda_{11} F(a, b, a + b - c + 1; 1 - z) + \lambda_{12} (1 - z)^{c-a-b} F(c - a, c - b, c - a - b + 1; 1 - z) \quad (76)$$

and

$$\begin{aligned} z^{1-c} F(a - c + 1, b - c + 1, 2 - c; z) &= \lambda_{21} F(a, b, a + b - c + 1; 1 - z) \\ &+ \lambda_{22} (1 - z)^{c-a-b} F(c - a, c - b, c - a - b + 1; 1 - z), \end{aligned} \quad (77)$$

where

$$\begin{aligned}\lambda_{11} &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \lambda_{21} = \frac{\Gamma(2-c)\Gamma(c-a-b)}{\Gamma(1-a)\Gamma(1-b)} \\ \lambda_{12} &= \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \quad \lambda_{22} = \frac{\Gamma(2-c)\Gamma(a+b-c)}{\Gamma(a-c+1)\Gamma(b-c+1)},\end{aligned}\tag{78}$$

one can expand the sets of solutions at $t_{in} = \infty$ in terms of the sets of solutions at $t_{out} = +\infty$. Using Eqs. (75-78), we have

$$\frac{|\beta_k|^2}{|\alpha_k|^2} = \frac{|\lambda_{12}|^2}{|\lambda_{11}|^2} = \frac{|\lambda_{22}|^2}{|\lambda_{21}|^2}.\tag{79}$$

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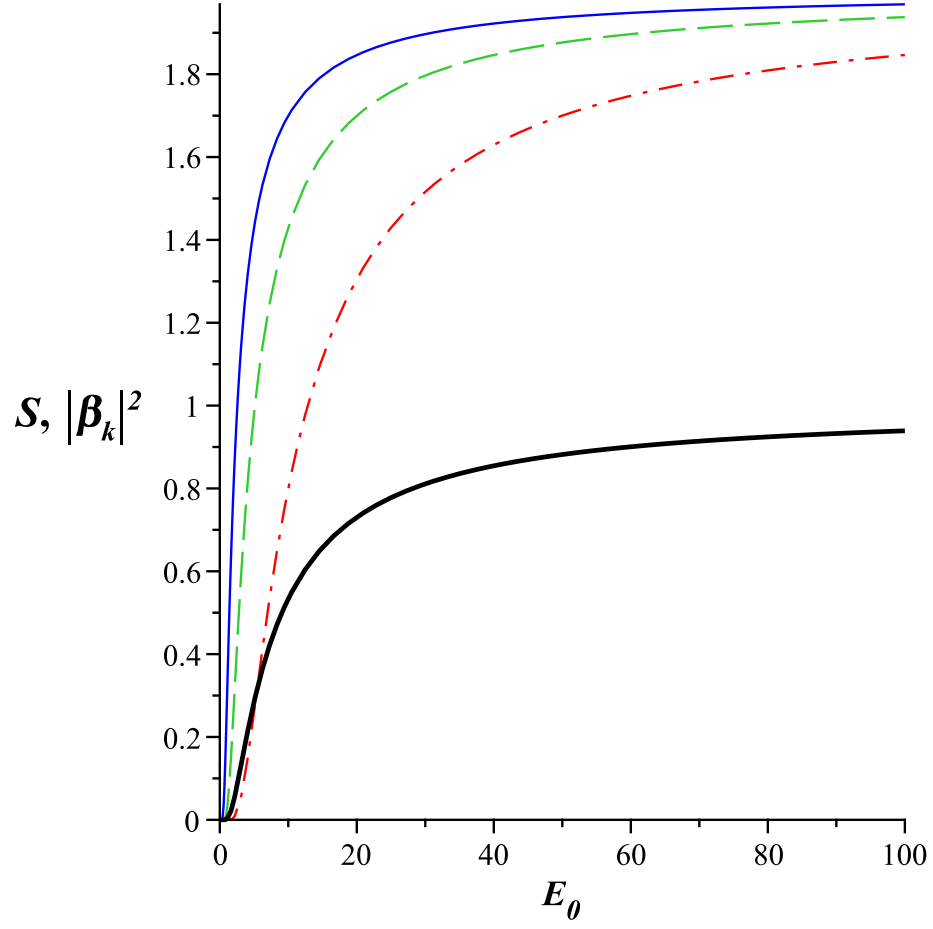


FIG. 1: (Color online) The von Neumann entropy for bosonic modes with respect to the electric field for $(q, k_\perp = 1)$ and certain values of mass: $m = 0$ (solid line), $m = 1$ (green dashed line), and $m = 2$ (red dash-dotted line). $|\beta|^2$ as a function of E_0 is represented by the solid thick black line for $m = 2$. In the large electric fields, $|\beta|^2$ tends to its maximum value, $|\beta_k|^2 \rightarrow 1$. Thus, regarding Eq. (30), entropy is a function of $|\beta_k|^2$ and at large values of the electric fields E_0 tends to a constant value ($S_k = 2$).

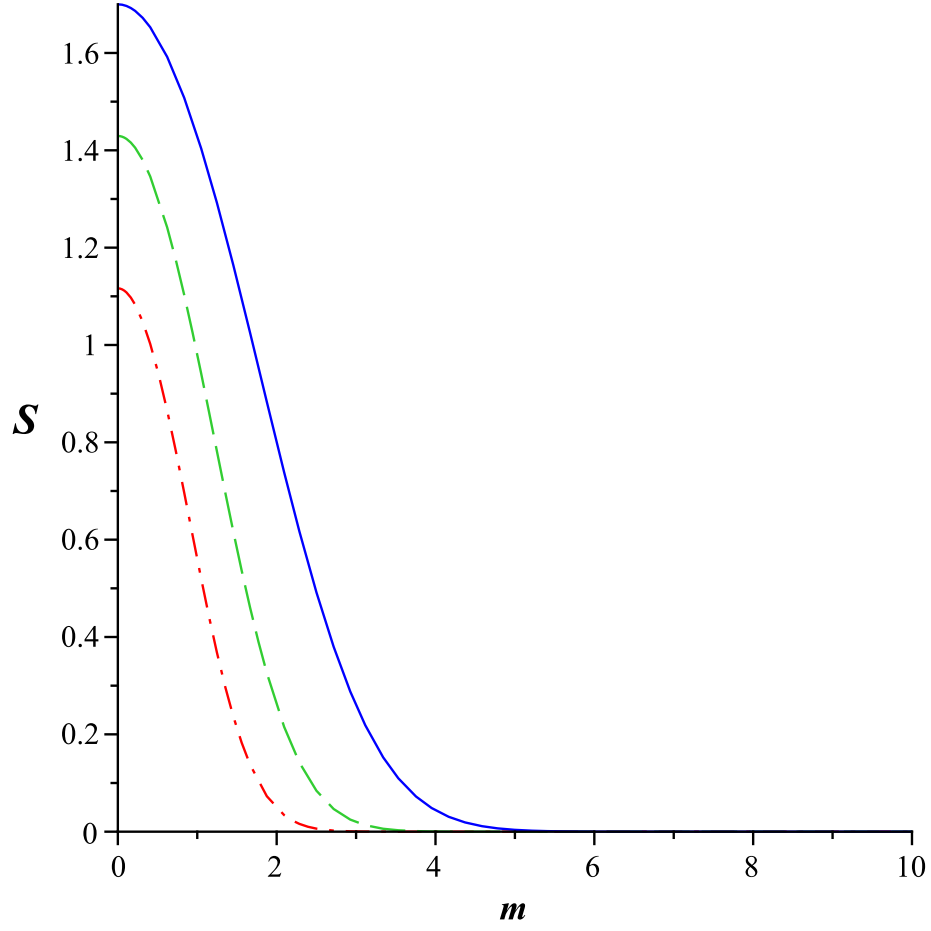


FIG. 2: (Color online) The von Neumann entropy for bosonic modes with respect to particle mass for $(q, k_{\perp} = 1)$ and certain values of the electric field: $E = 10$ (solid line), $E = 5$ (green dashed line), and $E = 3$ (red dash-dotted line). The maximum value of entanglement for fixed values of q , k_{\perp} and E_0 occurs in $m = 0$ which is obtained by substituting $|\beta_k|^2 = \exp(\frac{-\pi k_{\perp}^2}{2qE_0})$ in Eq. (30).

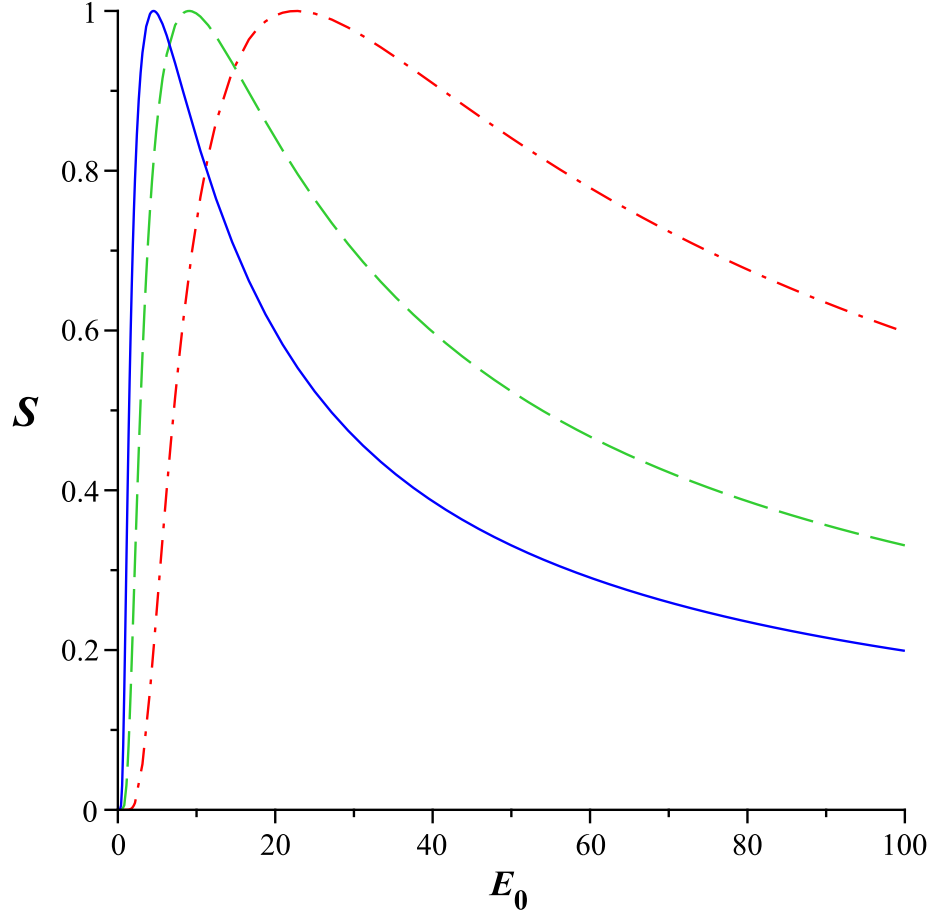


FIG. 3: (Color online) The von Neumann entropy for fermionic modes with respect to the electric field for $(q, k_{\perp} = 1)$ and certain values of the fermion mass: $m = 0$ (solid line), $m = 1$ (green dashed line), and $m = 2$ (red dash-dotted line). For very small or large electric fields, the value of $|\beta_k|^2$ tends to 0 or 1, respectively. Therefore, according to Eq. (58) the entropy has its minimum value, $S_{min} = 0$. The maximum value of entropy, $S_{max} = 1$, for fixed values of q , k_{\perp} and m , occurs in $E_0 = \frac{\pi(m^2 + k_{\perp}^2)}{q \ln(2)}$, which is equivalent to $|\beta_k|^2 = \frac{1}{2}$.

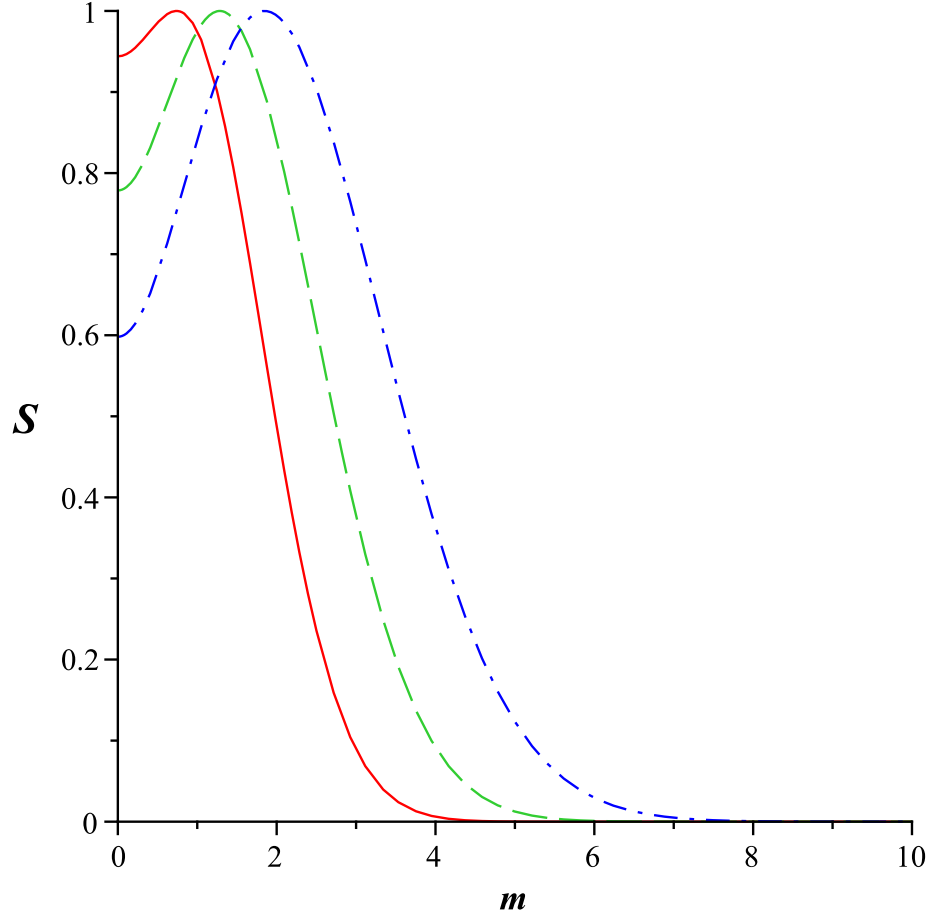


FIG. 4: (Color online) The von Neumann entropy for fermionic modes with respect to the particle mass for $(q, k_{\perp} = 1)$ and certain values of the electric field: $E = 7$ (red solid line), $E = 12$ (green dashed line), and $E = 20$ (blue dash-dotted line). The entanglement entropy is obtained by substituting $|\beta_k|^2 = \exp(\frac{-\pi k_{\perp}^2}{2qE_0})$ in Eq. (58), for massless particles. Large values of the electric fields correspond to large values of $|\beta_k|^2$ and therefore the small value of S_k . According to Eq. (58), for fixed values of E_0 , k_{\perp} and q , the maximum value of entropy is equal to one for $m^2 = \frac{E_0 q \ln(2)}{\pi} - k_{\perp}^2$.

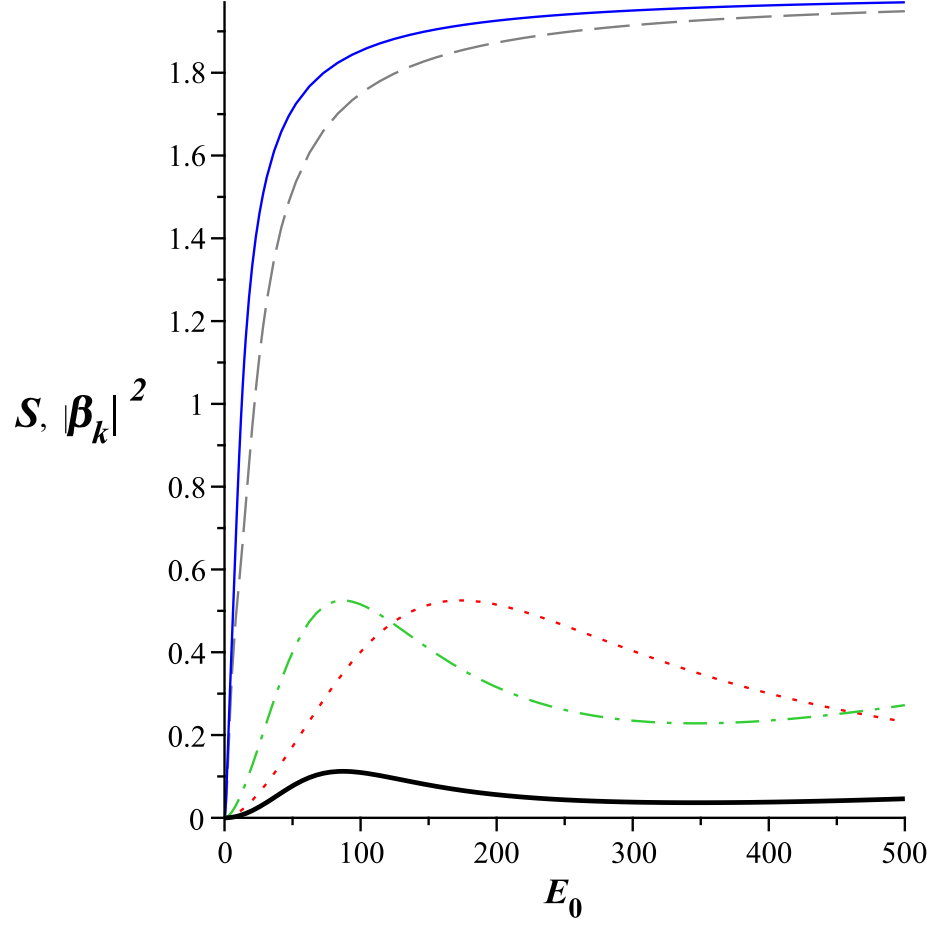


FIG. 5: (Color online) The von Neumann entropy for bosonic modes with respect to the amplitude of a pulsed electric field for $(m, q, k_{\perp}, k_z = 1)$ and certain values of τ : $\tau = .3$ (blue solid line), $\tau = .2$ (gray dashed line), $\tau = 0.02$ (green dash-dotted line), and $\tau = 0.01$ (red dotted line). Also, $|\beta_k|^2$ as a function of E_0 is represented by the lower solid thick black line for $\tau = 0.02$. The general behavior of $|\beta|^2$ is the same that of the von Neumann entropy. For small τ , $|\beta_k|^2$ has a local maximum and for fixed values of $m, q, K_{\perp}, k_z, \tau$, the related E_0 can be obtained numerically. By substituting $(|\beta_k|^2)_{max}$ in Eq. (69), The maximum value of S_k is obtained. The values of electric fields which maximize $|\beta_k|^2$ and S_k are equal. For large values of τ (blue solid line and gray dashed line), the pulsed electric field tends to constant electric field and therefore, the results are coincided to the constant one.

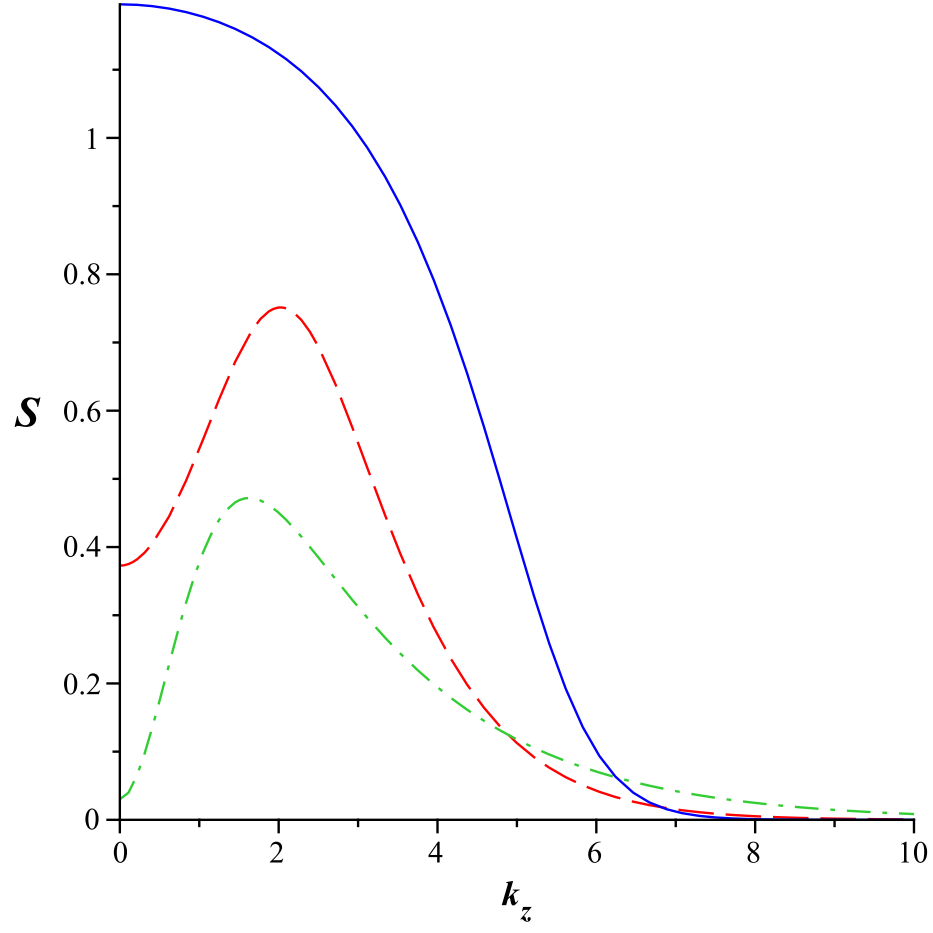


FIG. 6: (Color online) The von Neumann entropy for bosonic modes with respect to k_z for $(m, q, k_\perp = 1, E = 10)$ and certain values of τ : $\tau = 0.5$ (blue solid line), $\tau = 0.2$ (red dashed line), and $\tau = 0.1$ (green dash-dotted line). For fixed value of $(m, q, E_0, K_\perp, k_z)$, the entanglement entropy has local maximum. The related momentum can be obtained numerically.

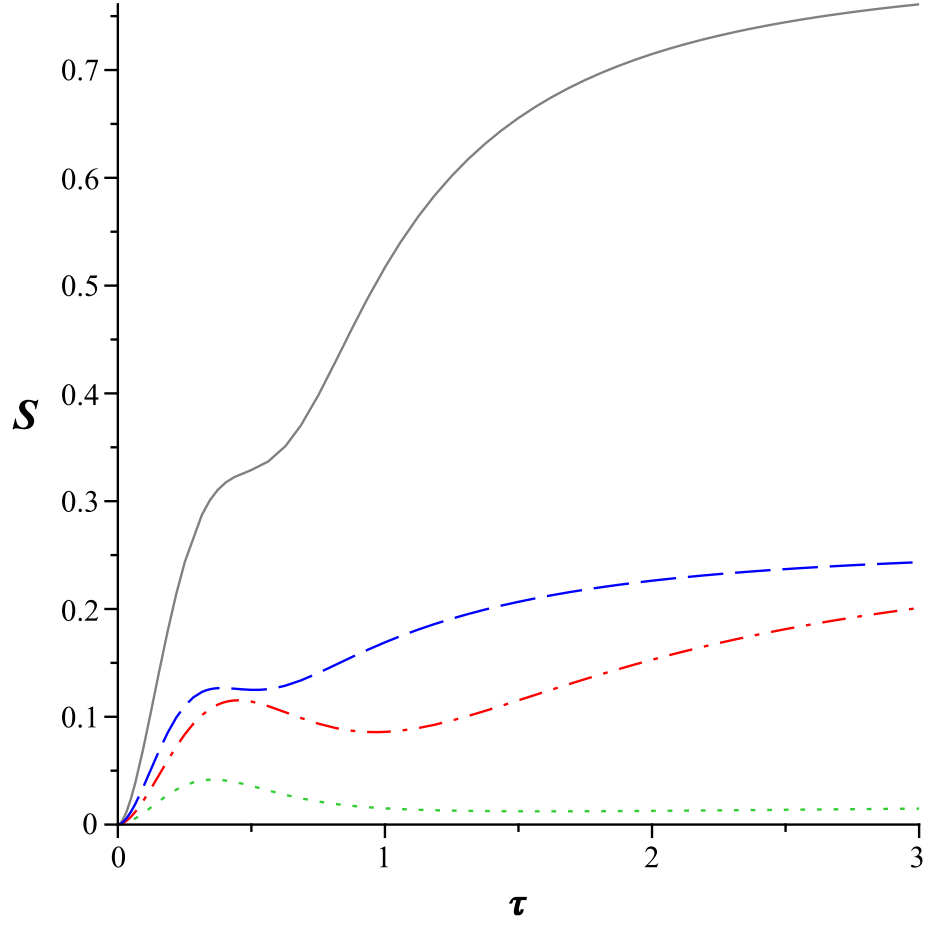


FIG. 7: (Color online) The von Neumann entropy for bosonic modes with respect to the width of pulse τ for $(m, q, k_z = 1)$ and certain values of E_0 and k_{\perp} : $E = 2, k_{\perp} = 0$ (gray solid line), $E = 2, k_{\perp} = 1$ (blue dashed line), $E = 1, k_{\perp} = 0$ (red dash-dotted line), and $E = 1, k_{\perp} = 1$ (the green dotted line). When $\tau \rightarrow \infty$, the pulsed electric field tends to the constant electric field and the results are coincided to the constant one. Therefore, the value of S_k depends on the fixed values of $(m, q, E, K_{\perp}, k_z)$.

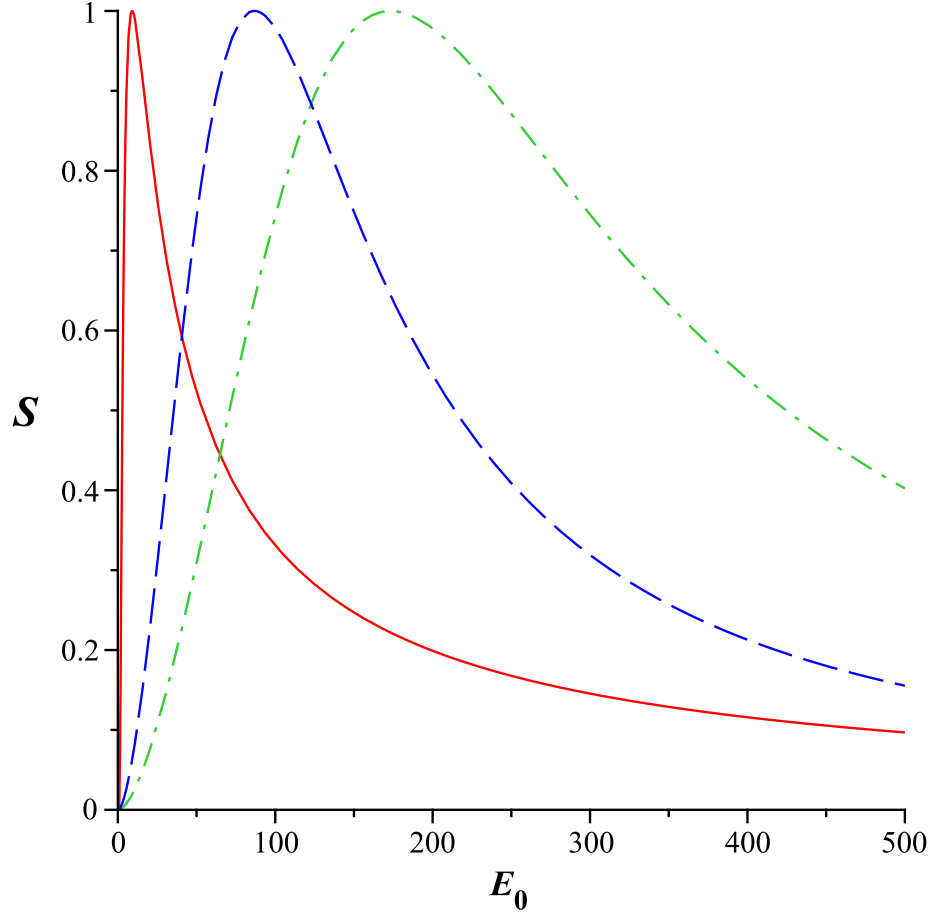


FIG. 8: (Color online) The von Neumann entropy for fermionic modes with respect to E_0 for $(m, q, k_z, k_\perp = 1)$ and some values of the τ , ($\tau = 2$)(red solid line), ($\tau = 0.02$)(blue dashed line), ($\tau = 0.01$) (green dash-dotted line). According to Eq. (73), in very small or large electric fields, $|\beta_k|^2$ tends to zero or one, respectively. Therefore, according to Eq. (74), the entropy has its minimum value, $S_{min} = 0$. The maximum value of entropy ($S_{max} = 1$), for fixed values of different parameters $(m, q, k_z, k_\perp, \tau)$ occurs at a specified electric field wherein $|\beta_k|^2 = 1/2$. The value of this electric field can be obtained numerically.

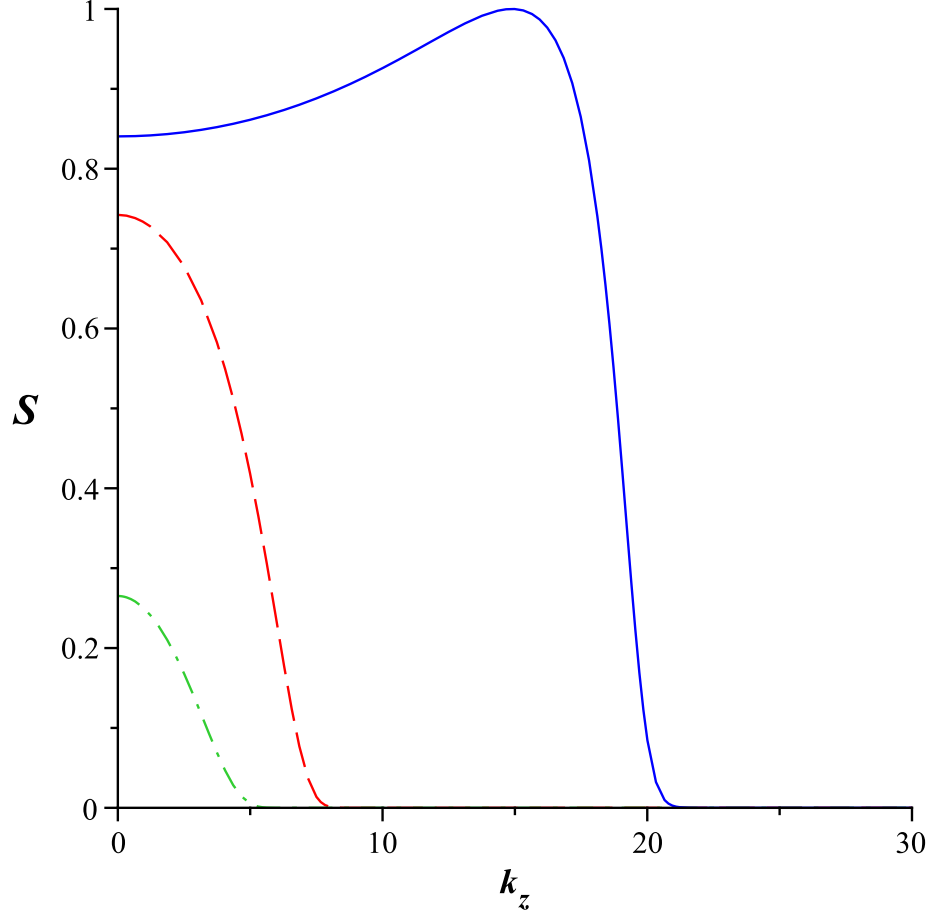


FIG. 9: (Color online) The von Neumann entropy for fermionic modes with respect to k_z for $(m, q, k_\perp = 1)$ and certain values of E_0 and τ : $E = 20, \tau = 1$ (blue solid line); $E = 4, \tau = 2$ (red dashed line); and $E = 2, \tau = 3$ (green dash-dotted line). The maximum value of entropy ($S_{max} = 1$), for fixed values of different parameters (m, q, \dots) , occurs wherein $|\beta_k|^2 = 1/2$. The related momentum can be obtained numerically. For $k_z = 0$; $0 < |\beta_k|^2 < 1$ and therefore $S_k \neq 0$. The value of S_k is obtained by substituting $|\beta_k|^2$ in Eq. (74).

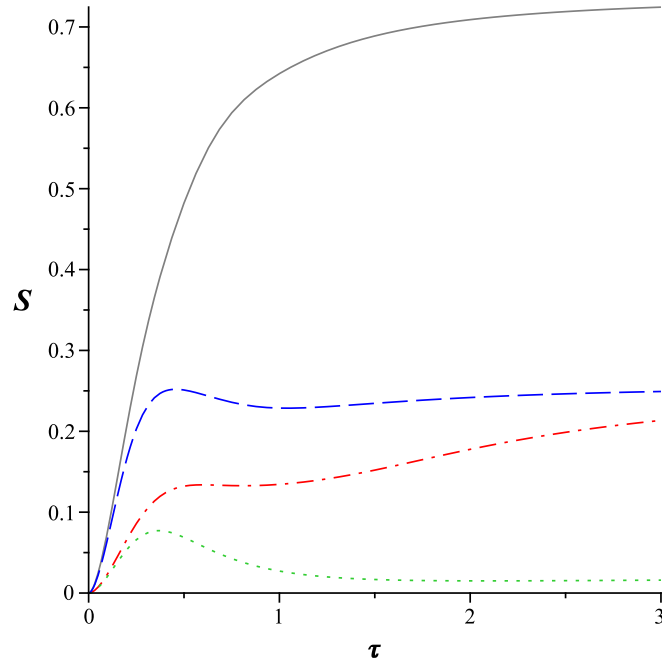


FIG. 10: (Color online) The von Neumann entropy for fermionic modes with respect to the width of pulse τ for $(m, q, k_z = 1)$ and some certain values of E_0 and k_\perp : $E = 2, k_\perp = 0$ (gray solid line); $E = 2, k_\perp = 1$ (blue dashed line); $E = 1, k_\perp = 0$ (red dash-dotted line); and $E = 1, k_\perp = 1$ (green dotted line). For $\tau = 0$; $|\beta_k|^2 = 0$ and therefore, $S_k = 0$. When $\tau \rightarrow \infty$, the pulsed electric field tends to the constant electric field and the results are coincided to the constant one. Therefore, the value of S_k depends on the fixed values of (m, q, E, K_\perp, k_z) .